G_2-geometry : torsion and curvature

M^7 - smooth manifold

satisfies - orientable and spinnable

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$w_1(TM) = 0 \qquad\qquad w_2(TM) = 0.$$

A G_2-structure on M exists $\Longleftrightarrow$ $w_1(M) = w_2(M) = 0$.

e.g. $\mathbb{R}^7$, S^7, Aloff-Wallach spaces, ...

$\rightsquigarrow$ $\mathbb{O}$, $G_2 = \mathrm{Aut}(\mathbb{O})$.

A G_2-structure on M $\longleftrightarrow$ $\varphi \in \Omega^3_+(M)$

φ_0 on $\mathbb{R}^7 = i^*\varphi_p$, $p \in M$.

$\{x^1, \dots, x^7\}$ local coordinates on M^7

$$\left(\frac{\partial}{\partial x^i} \lrcorner \varphi\right) \wedge \left(\frac{\partial}{\partial x^j} \lrcorner \varphi\right) \wedge \varphi = -6 B_{ij}\, dx^1 \wedge \dots \wedge dx^7$$

$$B_{ij} = B_{ji}$$

$$\boxed{g_{ij} = \frac{B_{ij}}{(\det B)^{1/9}}}$$

φ is a G_2-structure / positive 3-form / non-degenerate

$$\Leftrightarrow$$

g_{ij} is a Riemannian metric and a volume form.

In a nutshell,

$\varphi \in \Omega^3_+(M) \rightsquigarrow g$, dx, non-linear way

$\Omega^3_+(M)$ is an open subbundle of $\Omega^3(M)$.

Def^n A G_2-structure φ is called <u>torsion-free</u> if $\nabla^{\varphi}\varphi = 0$ where ∇^{φ} is the Levi-Civita connection of g_{φ}.

highly nontrivial problem to find φ's w/ $\nabla\varphi = 0$.

Lemma :- In local coordinates we have the following contraction identities:-

$$*\varphi = \psi \in \Omega^4(M).$$

1) $$\varphi_{ijk}\varphi_{abk} = \varphi_{ijk}\varphi_{abc}g^{kc} = g_{ia}g_{jb} - g_{ib}g_{ja} - \psi_{ijab}.$$

2) $$\varphi_{ijk}\varphi_{ajk} = 7g_{ia} - 6g_{ia} - 0 = 6g_{ia}$$

3) $\varphi_{ijk}\psi_{abck} = \varphi_{ijk}\psi_{abcd}\, g^{kd}$

$= g_{ia}\varphi_{jbc} + g_{ib}\varphi_{ajc} + g_{ic}\varphi_{abj}$

$- g_{ja}\varphi_{ibc} - g_{jb}\varphi_{aic} - g_{jc}\varphi_{abi}$.

$\square$

Decomposition of $\Omega^{\bullet}$ into irreducible G_2 representations

$\Omega^2 = \Omega^2_7 \oplus_{\perp} \Omega^2_{14}$, 7 and 14 are pointwise dimensions

$\Omega^3 = \Omega^3_1 \oplus_{\perp} \Omega^3_7 \oplus_{\perp} \Omega^3_{27}$

$\Omega^4 = *\Omega^3 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}$

$\Omega^5 = *\Omega^2 = \Omega^5_7 \oplus \Omega^5_{14}$ . . .

$$\Omega^2_7 = \{ X \lrcorner \varphi \mid X \in \Gamma(TM) \} = \{ \beta \in \Omega^2 \mid *(\varphi \wedge \beta) = -2\beta \}$$

$$\Omega^2_{14} = \{ \beta \in \Omega^2 \mid \beta \wedge \psi = 0 \} = \{ \beta \in \Omega^2 \mid *(\varphi \wedge \beta) = \beta \}$$

$$P : \Omega^2 \longrightarrow \Omega^2$$

$$P(\beta) = *(\varphi \wedge \beta) \qquad \text{if } \beta = \frac{1}{2} \beta_{ij}\, dx^i \wedge dx^j$$

$$\Rightarrow (P\beta)_{ab} = \frac{1}{2} \beta_{ij} \psi_{abij} = \frac{1}{2} \beta_{ij} \psi_{abcd}\, g^{ic} g^{jd}$$

P is a self-adjoint map $\Rightarrow$ diagonalizable w/ real eigenvalues.

exercise.

$$\langle P\alpha, \beta \rangle = \langle \alpha, P\beta \rangle.$$

$$(P^2\beta)_{ab} = \frac{1}{2} \psi_{abij} (P\beta)_{ij}$$

$$= \frac{1}{4} \psi_{abij} \beta_{mn} \psi_{ijmn}$$

$$= \frac{1}{4} \beta_{mn} \left(4 g_{am} g_{bn} - 4 g_{an} g_{bm} - 2 \psi_{abmn} \right)$$

$$= 2\beta_{ab} - (P\beta)_{ab}.$$

$\Longrightarrow$

$$P^2 = 2I - P \Rightarrow (P+2I)(P-I) = 0.$$

$\therefore$ the eigenvalues for P are -2 and $+1$.

$$\beta_{ij} \in \Omega^2_7 \Longleftrightarrow \beta_{ij} = X_m \varphi_{mij} \Longleftrightarrow \frac{1}{2}\psi_{abij}\beta_{ij} = -2\beta_{ab}.$$

$$\beta_{ij} \in \Omega^2_{14} \Longleftrightarrow \beta_{ij}\varphi_{ijm} = 0 \Longleftrightarrow \frac{1}{2}\psi_{abij}\beta_{ij} = +1\beta_{ab}.$$

Explicit description of the decomposition of $\Omega^3(M)$

$A \in \Gamma(T^*M \otimes TM)$, i.e. A is a $(1,1)$-tensor.

$$e^{At} \in GL(T_pM)$$

$$e^{At} \cdot \varphi = \frac{1}{3!} \varphi_{ijk} (e^{At} dx^i) \wedge (e^{At} dx^j) \wedge (e^{At} dx^k)$$

<u>Define</u>:-

$$(A \diamond \varphi) = \frac{d}{dt}\Big|_{t=0} (e^{At} \cdot \varphi)$$ infinitesimal action of $GL(7,\mathbb{R})$ on φ.

given any 2-tensor A on M

$$\boxed{(A \diamond \varphi)_{ijk} = A_{ip}\varphi_{pjk} + A_{jp}\varphi_{ipk} + A_{kp}\varphi_{ijp}.}$$

Linear algebra:-

$$\Gamma(T^*M \otimes T^*M) \cong \underbrace{\Omega^0 \oplus S^2_0}_{\text{symmetric}} \oplus \underbrace{\Omega^2}_{\text{skew-symmetric}}$$

$$= \Omega^0 \oplus S^2_0 \oplus \Omega^2_7 \oplus \Omega^2_{14}$$

$\therefore$

$$A = \frac{1}{7}(\mathrm{tr}\,A)g + A_0 + A_7 + A_{14}$$

we have a linear map.

$$A \longmapsto A \diamond \varphi$$

The kernel of the map $A \longmapsto A \diamond \varphi$ is precisely Ω^2_{14}.

$A \mapsto A \diamond \varphi$ maps the space $\Omega^0 \oplus S^2_0 \oplus \Omega^2_7$ isomorphically onto $\Omega^3_1 \oplus \Omega^3_{27} \oplus \Omega^3_7$.

$$A \diamond \varphi = \underbrace{\frac{3}{7}(\mathrm{tr}\,A)\varphi}_{\Omega^3_1} + \underbrace{A_0 \diamond \varphi}_{\in\, \Omega^3_{27}} + \underbrace{A_7 \diamond \varphi}_{\in\, \Omega^3_7}$$

This gives me

$$A = A_0 + \frac{\operatorname{tr} A}{7} + A_7 + A_{14}$$

$$\Omega^3_1 = \left\{ \frac{\operatorname{tr} A}{7} \diamond \varphi \;\middle|\; A \in \Omega^0, \text{ i.e. } \frac{\operatorname{tr} A}{7} \text{ w/ } A \text{ a 2-tensor} \right\}$$

$$\Omega^3_7 = \{ A_7 \diamond \varphi \}$$

$$\Omega^3_{27} = \{ A_{27} \diamond \varphi \}.$$

Torsion of a G_2-structure

Thm Let $X \in \Gamma(TM)$. Then the 3-form $\nabla_X \varphi$, which a priori can lie in $\Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}$, actually lies in the subspace Ω^3_7.

In other words, $\nabla\varphi \in \Gamma(T^*M \otimes \Lambda^3_7(T^*M))$.

Proof. Suppose $\gamma \in \Omega^3(M)$ arbitrary.
we know, $\gamma = A \diamond \varphi$ for some unique
$A = \frac{1}{7}(\operatorname{tr} A) g + A_0 + A_7$.

$$\langle \gamma, \nabla_X \varphi \rangle = \langle A \diamond \varphi, \nabla_X \varphi \rangle$$

$$= \frac{1}{6} (A \diamond \varphi)_{ijk} (\nabla_X \varphi)_{ijk} = \frac{1}{6} (A \diamond \varphi)_{ijk} (\nabla_X \varphi)_{abc} g^{ia} g^{jb} g^{kc}.$$

$$= \frac{1}{6} (A_{ip} \varphi_{pjk} + A_{jp} \varphi_{ipk} + A_{kp} \varphi_{ijp}) X_m \nabla_m \varphi_{ijk}$$

$$= \frac{1}{2} A_{ip} \varphi_{pjk} X_m \nabla_m \varphi_{ijk}$$

$$= \frac{1}{2} A_{ip} X_m \underbrace{\varphi_{pjk} \nabla_m \varphi_{ijk}}_{\text{skew in } i \text{ and } p.} \quad \text{—— } (1)$$

recall $\quad \varphi_{ijk} \varphi_{pjk} = 6 g_{ip}$

differentiate both sides :-

$$(\nabla_m \varphi_{ijk}) \varphi_{pjk} + \varphi_{ijk} (\nabla_m \varphi_{pjk}) = 6 \nabla_m g_{ip}$$

$$\Rightarrow (\nabla_m \varphi_{ijk}) \varphi_{pjk} = - (\nabla_m \varphi_{pjk}) \varphi_{ijk}$$

$\therefore$ only the skew part of A will give any non-zero contribution

$$\Rightarrow \langle A \diamond \varphi, \nabla_X \varphi \rangle = \langle A_7 \diamond \varphi, \nabla_X \varphi \rangle$$

$\Rightarrow$ $\nabla_X \varphi$ is orthogonal to $\Omega^3_{1 \oplus 27}$

$\Rightarrow$ $\nabla_X \varphi \in \Omega^3_7$.

$\blacksquare$

$$\because \quad \nabla_X \varphi \in \Omega^3_7 = \{ X \lrcorner \psi \mid X \in \Gamma(TM) \}$$

$$\therefore \quad \boxed{\nabla_X \varphi = T(X) \lrcorner \psi}$$

where $T(X) \in \Gamma(TM) \Rightarrow T$ is a 2-tensor.

<u>Defn</u> The 2-tensor T is called the full-torsion tensor of φ and is explicitly given in local coordinates as

$$T_{ij} = \frac{1}{2}(\nabla_i \varphi_{abc}) \psi_{jabc}$$

$$\Updownarrow$$

$$\boxed{\nabla_m \varphi_{ijk} = T_{mp}\, \psi_{pijk}}$$

$$\boxed{\varphi \text{ is torsion-free} \iff \nabla\varphi = 0 \iff T = 0.}$$

$$T = \underbrace{\overbrace{\frac{\operatorname{tr} T}{7}}^{T_1} g + T_{27}}_{\text{symmetric}} + \underbrace{T_7 + T_{14}}_{\text{skew-symm.}}$$

$$\therefore \quad T = T_1 + T_{27} + T_7 + T_{14}$$

T_i's are called **intrinsic torsion forms**.

$T = 0 \iff T_i = 0 \quad \forall\ i = 1, 27, 7$ and 14.

Thm/ (Fernández-Gray '86)

Let φ be a G_2-structure on M. Then φ is torsion-free $\iff d\varphi = 0$ and $d^*\varphi = 0$, i.e. φ is both closed and co-closed.

$\nabla\varphi \Longleftrightarrow d\varphi = 0$ and $d^*\varphi = 0$ ($d\psi = 0$)

$\nabla\varphi = 0$. $\Longrightarrow$ obvious. $d^*\varphi = {*d*\varphi} = 0 \Rightarrow d*\varphi = 0 \Rightarrow d\psi = 0$.

d is the skew-symmetrization of ∇.

$$d^*\gamma = -\nabla_i \gamma_{ijk}$$

$$d\varphi \in \Omega^4 = \Omega^4_{1\oplus 7\oplus 27}$$

$$d^*\varphi \in \Omega^2 = \Omega^2_{7\oplus 14}.$$

$d\varphi$ and $d^*\varphi$ are linear in $\nabla\varphi \leadsto d\varphi$ and $d^*\varphi$ are linear in T.

Schur's lemma $\Rightarrow$ independent components of $d\varphi$ and $d^*\varphi$ = components of T in $\Omega^2_7, \Omega^2_{14}, \Omega^4_{1\oplus 7+27}$

components are T_1, T_7, T_{14}, T_{27}.

$\therefore$ if $d\varphi = 0$ and $d^*\varphi = 0$. $\Rightarrow$ $T_1 = 0 = T_7 = T_{14} = T_{27}$

$\Rightarrow$ $T = 0$ $\Rightarrow$ $\nabla\varphi = 0$.

$\boxed{}$

In other words: question of finding Torsion-free G_2-structures

$\Leftrightarrow$ $\nabla\varphi = 0$

$\Leftrightarrow$ $d\varphi = 0$ and $d^*\varphi = 0$.

<u>Relationship b/w curvature of g_φ and T_φ.</u>

<u>Thm.</u> (Karigiannis '09) [G_2-Bianchi identity]

Let (M^7, φ). g_φ, T, torsion.

$$\nabla_i T_{jk} - \nabla_j T_{ik} = T_{ia} T_{jb} \varphi_{abk} + \frac{1}{2} R_{ijab} \varphi_{abk}$$
$$= \left[T_{ia} T_{jb} + \frac{1}{2} R_{ijab} \right] \varphi_{abk}.$$

$$= \left[T_{ia}T_{jb} + \frac{1}{2}R_{ijab}\right]\varphi_{mnk}\, g^{am} g^{bn}.$$

R_{ijab} are the components of the $(4,0)$ Riemann curvature tensor.

Proof :-
$$T_{jk} = \frac{1}{24}\left(\nabla_j \varphi_{abc}\right)\psi_{kabc}$$

$$\Rightarrow \quad \nabla_i T_{jk} = \frac{1}{24}\left(\nabla_i \nabla_j \varphi_{abc}\right)\psi_{kabc} + \frac{1}{24}\left(\nabla_j \varphi_{abc}\right)\nabla_i \psi_{kabc}$$

$$\nabla_j T_{ik} = \frac{1}{24}\left(\nabla_j \nabla_i \varphi_{abc}\right)\psi_{kabc} + \frac{1}{24}\left(\nabla_i \varphi_{abc}\right)\nabla_j \psi_{kabc}.$$

Ricci identities.

$$\therefore \nabla_i T_{jk} - \nabla_j T_{ik} = \frac{1}{24}\left(\overbrace{\nabla_i \nabla_j \varphi_{abc} - \nabla_j \nabla_i \varphi_{abc}}\right)\psi_{kabc} + \frac{1}{24}\left(\left(\nabla_j \varphi_{abc}\right)\nabla_i \psi_{kabc} - \underset{i \leftrightarrow j}{\quad}\right)$$

$\boxed{4}$

<u>Com.</u>: The Ricci curvature of g_φ

$$R_{jk} = (\nabla_i T_{jp} - \nabla_j T_{ip})\varphi_{ipk} - T_{jp}T_{pk}$$
$$+ (\operatorname{Tr} T) T_{jk} - T_{jl} T_{pq} \psi_{pqlk}.$$

<u>Thm.</u> [Bonan '1966].

(M^7, φ) w/ φ torsion-free are Ricci-flat.

φ TF $\Rightarrow \nabla\varphi = 0 \Rightarrow T = 0 \Rightarrow Ric = 0.$

M w/ compact,

Torsion-free G_2-structure are the only sources of Ricci-flat manifolds in odd dimensions, yet.

<u>Thm.</u> Let (M^7, φ) be a manifold w/ a G_2-structure.

Then TFAE:

1) φ is torsion-free.

2) $\mathrm{Hol}(g) \subseteq G_2$.

3) $\nabla\varphi = 0$.

4) $d\varphi = d^*\varphi = 0$ on M.

Prop Let (M^7, φ, g) be a compact manifold w/ torsion-free φ. Then $\mathrm{Hol}(g) = G_2 \Longleftrightarrow \pi_1(M)$ is finite.

$\mathrm{Hol}(g)$ can be $\{id\}$, $SU(2)$, $SU(3)$ or G_2.

Suppose M is compact and $\nabla\varphi = 0$. $\Longrightarrow$

$\mathrm{Ric}(g) \equiv 0$.

$\Downarrow$ Cheeger–Gromoll splitting theorem.

$\Downarrow$ compact, Ricci flat $g \Longrightarrow$

M has a finite cover isometric to $T^k \times N$ where T^k is the k-dim flat torus and N is compact and simply-connected.

$\boxed{}$ M^7 has finite cover isometric to $\mathbb{T}^k \times N$ w/ N simply-connected.

$\Rightarrow$ $\pi_1(M) \cong F \ltimes \mathbb{Z}^k$

F is a finite group.

$k=7$ $M \cong \mathbb{T}^k$, $Hol(g) = \{1\}$.

$k=3$, $M \cong \mathbb{T}^3 \times N^4$, $Hol(g) = SU(2)$.

$k=1$ $M \cong \mathbb{T}^1 \times N^6$, $Hol(g) = SU(3)$

$k=0$, $M = N^7 \Rightarrow Hol(g) = G_2$

$\Downarrow$

$\pi_1(M) = F \ltimes \mathbb{Z}^0$

$= F$ which is finite.

$\boxed{}$

Today :-

* saw explicit decomposition of $\Omega^{\bullet}$ in terms of data from φ.

* explicitly description of $T \simeq \nabla\varphi$ and the intrinsic torsion forms.

* $\nabla\varphi = 0 \iff d\varphi = 0$ and $d^{*}\varphi = 0$.

* $\nabla\varphi = 0 \implies Ric = 0$.

* compact (M^7, φ), $Hol(g) = G_2 \iff |\pi(M)| < \infty$

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